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Calculation of Eigenvalue and Eigenvector Derivatives for Algebraic Flutter and Divergence Eigenproblems

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A single procedure is presented for the determination of eigenvalue and eigenvector derivatives of general eigensystems, including those arising in flutter and divergence analysis. The method requires only the knowledge of the eigensolution under consideration and no information about the transposed problem. In the procedure, the eigenproblem is not presented as such, but as a nonlinear algebraic system of equations with the eigenvalue and the eigenvector as unknowns and the eigenderivatives coming from the solution of a linear system of equations that is determined trivially by differentiating the nonlinear equations with respect to a structural parameter. The technique can be extended easily to derivatives of any order and always requires solutions of the same linear system with different right-hand sides. The coefficient matrix of this linear system always maintains the structure of the matrix of the eigenproblem and three methods of solution are presented to take full advantage from its symmetry and sparseness, if present. Application to flutter and divergence problems is emphasized, as the basic idea of the method unifies the determination of a solution and of its derivatives in one single approach.

Introduction

THE solution of linear and quadratic eigenproblems is required for many problems of stability and response analysis of linear structures. To cite only a few: the determination of vibration modes and frequencies for steady and rotating structures; linearized buckling and the response and stability of systems with external forces linearly depending upon motion parameters such as those arising in aeroelastic analysis, and typically in the flutter and divergence problem.

Therefore, substantial research has been devoted to the development of numerical methods capable of efficiently solving larger and larger problems of that kind by digital computers.[1-7] Now, as the search for optimal design of structures in which response and stability problems pose major constraints is becoming mandatory, it is often desirable to be able to estimate efficiently the sensitivity of the available designs to changes in system parameters.

This calls immediately for efficient methods for evaluating the derivatives of eigenvalues and eigenvectors. The use of such derivatives is manifold, according to the degree of automation of the adopted design procedure. It is very helpful in contributing physical understanding and insight to the designer who is using the computer as a tool for the analysis. Also it is essential to finding gradients in optimization procedures such as automatic structural design with dynamic and/or stability constraints, optimum design of control systems, and system identification.

The availability of derivatives of first, second, and sometimes higher orders is often sufficient in estimating significant design performance over a relatively wide range of values of structural parameters by the use of perturbation techniques. Many methods are available to calculate derivatives for practically any eigenproblem with arbitrary matrices.[8-14] All of these methods allow an easy calculation of any eigenvalue derivative, since they require only the knowledge of the eigenvalue itself and the corresponding right and left eigenvectors. But many of them, when used to calculate any single eigenvector derivative, are somewhat inefficient as the knowledge of all the eigenvectors is needed.

The most recent works[12-14] have contributed very much in enhancing efficiency, as they allow finding the derivative of a particular eigenvector only with the knowledge of the corresponding eigenvalue, its derivatives, and its right and left eigenvectors, but some of them tend to destroy the sparsity of the matrices involved.

It is to be noted that the quadratic eigenproblem is seldom explored, but the methods just referenced seem to be readily applicable to this problem, too.

Finally, it is important to remark that the determination of the stability boundaries in divergence and flutter problems gives rise to an eigenvalue formulation that differs from the algebraic one since the eigenvalues, generally frequency and/or speed, may not appear in polynomial form due to the presence of aerodynamic matrices. Therefore, the methods used to calculate eigenderivatives in these problems require a specialized development. An alternate approach is taken in this paper by considering the eigenproblem as a nonlinear system of equations once a normalization condition of the eigenvectors has been added.

This idea, first presented in Ref. 15 only as a method for solving any eigenproblem directly, affords the basis for the development of a powerful and unifying technique for the eigenderivative calculation, while in its original purpose it is not as efficient as any of the methods generally used for eigensolution, except in the flutter case as shown in Refs. 16 and 17.

The technique applies straightforwardly to any type of problem and does not separate the calculation of the eigenvalue derivative from that of the eigenvector, but it allows the determination of them all at once, and requires only the eigenvalue and the right eigenvector under consideration.

The sparseness of component matrices is maintained as far as possible and is especially efficient when the derivative of a whole eigensolution is required.

The application is limited to the class of eigenproblems with a full set of distinct eigenvalues, but this is not believed to be a severe limitation as most of the practical cases are of this kind. Moreover, it must be noted that the case of multiple eigensolution is rarely treated in the literature.[15,18]

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The Method

The concept of the method will be explained extensively for a linear eigenproblem expressed in canonical form

$$([A]-\lambda[I])\{x\}=0 \tag{1}$$

where $[A]$ is an arbitrary matrix of order n.

This is done only in order to simplify the presentation as it requires a minimum of notation in the formulas and all the concepts involved can be trivially extended to all the eigenproblems that will be considered in the paper.

The basic idea is to consider Eq. (1) plus a normalization condition for $\{x\}$; for instance,

$$\{x\}^T\{x\}=1 \tag{2}$$

as a system of $n+1$ nonlinear equations in $n+1$ unknowns, the components of $\{x\}$ and λ, that is written as:

$$\begin{cases} ([A]-\lambda[I])\{x\}=0 \\ \{x\}^T\{x\}=1 \end{cases} \tag{3}$$

It can be noted that another way of applying this idea is to fix the value of a component of $\{x\}$ and to consider Eq. (1) directly as a nonlinear problem with $n-1$ components of $\{x\}$ and λ as unknowns.

The formulation of Eq. (3) is preferable, since if $[A]$ is a symmetric matrix, a significant saving in the following calculation will result.

If we indicate the derivative with respect to an arbitrary parameter d_i by a comma followed by the index i, by simply differentiating Eq. (3) for a particular solution $\{x_k\}$, λ_k with respect to d_i, we have the following linear system of equations with $\{x_k\}_{,i}$ and $\lambda_{k,i}$ as unknowns

$$\left[\begin{array}{c|c} ([A]-\lambda_k[I]) & -\{x_k\} \\ \hline -\{x_k\}^T & 0 \end{array}\right] \left\{\begin{array}{c} \{x_k\}_{,i} \\ \hline \lambda_{k,i} \end{array}\right\} = -\left\{\begin{array}{c} [A]_{,i}\{x_k\} \\ \hline 0 \end{array}\right\} \tag{4}$$

In this way the desired derivatives come from the solution of the preceding linear system without any need of calculating either the left eigenvector of λ_k or any of the remaining right eigenvectors.

It is important to prove that if $[A]$ possesses n-distinct eigenvalues and eigenvectors, the full system, Eq. (4), is well behaved. If $[Y]$ is a matrix whose columns are the n-independent left eigenvectors, i.e., the n solutions of

$$([A]^T-\lambda[I])\{y\}=0 \tag{5}$$

and $[X]$ is the matrix with the right eigenvectors, i.e., the n solutions of Eq. (1), as columns, it is well known that[15]

$$[Y]^T[X]=[s_i] \tag{6a}$$

and

$$[Y]^T[A][X]=[\lambda_i s_i] \tag{6b}$$

where $[s_i]$ and $[\lambda_i s_i]$ are diagonal matrices with s_i and $\lambda_i s_i$ as components. All the s_i are always different from zero.

As $[X]$ is nonsingular, we can uniquely define $\{x_k\}_{,i}$ as a linear combination of the n right eigenvectors by the transformation

$$\{x_k\}_{,i}=[X]\{t\} \tag{7}$$

Substituting this equation into Eq. (4), premultiplying its upper partitions by $[Y]^T$, and remembering Eqs. (6a) and (6b), we transform Eq. (4) into the following:

$$\left[\begin{array}{ccc|c} (\lambda_1-\lambda_k)s_1 & & & 0 \\ & \ddots & & \vdots \\ & (\lambda_{k-1}-\lambda_k)s_{k-1} & & 0 \\ & 0 & & -s_k \\ & & (\lambda_{k+1}-\lambda_k)s_{k+1} & 0 \\ & & \ddots & \vdots \\ & & (\lambda_n-\lambda_k)s_n & 0 \\ \hline r_1 \ldots & r_{k-1} & -1 r_{k+1} \ldots r_n & 0 \end{array}\right] \left\{\begin{array}{c} t_1 \\ \vdots \\ t_{k-1} \\ t_k \\ t_{k+1} \\ \vdots \\ t_n \\ \hline 0 \end{array}\right\}$$

$$= -\left\{\begin{array}{c} [Y]^T[A]_{,i}\{x_k\} \\ \\ \\ \\ \hline 0 \end{array}\right\} \tag{8a}$$

where

$$r_l=-\{x_k\}^T\{x_l\} \quad (l=1\ldots n) \tag{8b}$$

which clearly has a unique solution. Thus, the full system of Eq. (4) because of Eq. (8) can be uniquely solved, too. Looking for the best way of drawing advantages from the sparseness and symmetry of $[A]$, we devise three methods for solving Eq. (4).

1) Method I: In the case of a fully populated matrix $[A]$ of any type or when a solution method is available that can handle arbitrarily sparse matrices, Eq. (4) is directly solved once a suitable pivoting technique has been provided.

2) Method II: In the case of particular sparse matrices, for instance, symmetric or unsymmetric banded matrices, the pivoting over all the $n+1$ equations can destroy their structure. In order to devise an efficient method for these cases, we rewrite Eq. (4) as:

$$\left[\begin{array}{c|c|c} ([A]-\lambda_k[I])^* & \{c\}^* & -\{x_k\} \\ \hline \{r\}^T & 0 & -\bar{x}_k \\ \hline -\{x_k\}^T & 0 & 0 \end{array}\right] \times \left\{\begin{array}{c} \{x_k\}_{,i} \\ \hline \bar{x}_{k,i} \\ \hline \lambda_{k,i} \end{array}\right\} = -\left\{\begin{array}{c} [A]_{,i}\{x_k\} \\ \hline \bar{n} \\ \hline 0 \end{array}\right\} \tag{9}$$

where $\{r\}$ is a vector duplicating a properly chosen row l of $([A]-\lambda_k[I])$, $\bar{x}_k$, $\bar{x}_{k,i}$, and $\bar{n}$ are the lth components of $\{x_k\}$, $\{x_k\}_{,i}$, and $[A]_{,i}\{x_k\}$, respectively, $\{c\}^*$ is a vector with all the components zero except the lth one which is a negative number with a very large modulus (10^{30}), and the asterisk indicates that $([A]-\lambda_k[I])$ has been modified by adding the modulus of the same large number to the lth element of its diagonal.

By these modifications we have only introduced in a simple way the equation

$$x_{k,l}^{(l)}=\bar{x}_{k,i}$$

$x_{k,i}^{(l)}$ being the lth component of $\{x_k\}_{,i}$ while maintaining the original topology of $[A]$ in $([A]-\lambda_k[I])^*$.

Now, Eq. (9) can be rewritten explicitly as:

$$\{x_k\}_{,i}=\{u\}\bar{x}_{k,i}+\{v\}\lambda_{k,i}+\{t\} \tag{10a}$$

$$\{r\}^T\{x_k\}_{,i}-\bar{x}_k\lambda_{k,i}=-\bar{n} \tag{10b}$$

$$\{x_k\}^T\{x_k\}_{,i}=0 \tag{10c}$$

with

$$\{u\}=-([A]-\lambda_k[I])^{*-l}\{c\}^* \tag{11a}$$

$$\{v\}=([A]-\lambda_k[I])^{*-l}\{x_k\} \tag{11b}$$

$$\{t\}=-([A]-\lambda_k[I])^{*-l}[A]_{,i}\{x_k\} \tag{11c}$$

where the inverse of the starred matrix exists if the choice of l has been made properly.

In practice, this inverse matrix is never explicitly calculated. On the contrary, it is factored into its upper and lower triangular factors, as this operation can conserve the structure of the matrix in the best way, and $\{u\}$, $\{v\}$, and $\{t\}$ are obtained by a forward-backward solution process on $-\{c\}^*$, $\{x_k\}$, and $-[A]_{,i}\{x_k\}$. By substituting Eq. (10a) into Eqs. (10b) and (10c), we can obtain $\bar{x}_{k,i}$ and $\lambda_{k,i}$ from the solution of the following two linear equations:

$$\begin{bmatrix} \{r\}^T\{u\} & -\bar{x}_k+\{r\}^T\{v\} \\ \{x_k\}^T\{u\} & \{x_k\}^T\{v\} \end{bmatrix} \times \begin{Bmatrix} \bar{x}_{k,i} \\ \lambda_{k,i} \end{Bmatrix} = -\begin{Bmatrix} \bar{n}+\{r\}^T\{t\} \\ \{x_k\}^T\{t\} \end{Bmatrix} \tag{12}$$

Then by back-substituting the two values obtained into Eq. (10a), the problem is fully solved. Clearly, the most expensive part of the method, i.e., the factorizing of $([A]-\lambda_k[I])^*$, will be performed as efficiently as possible by the most appropriate of the methods available for the particular problem at hand.

The problem of the proper choice of the index l in order to maintain $([A]-\lambda_k[I])^*$ as well conditioned as possible, is the same as that presented in Ref. 14 for the calculation of an eigenvector derivative. There it is shown that for symmetric matrices, l is the index of the component of $\{x_k\}$ with the largest modulus, while for unsymmetric matrices it depends also on the components of the left eigenvectors.

Thus, the choice is made a priori for the symmetric case while in the unsymmetric one, since we do not know the left eigenvector, the choice of the row is made during the factorization process which employs pivoting to insure numerical stability, but only on part of the columns to preserve sparseness.

During this process when we meet a pivot that can be retained as zero, we add the large number to it and record the row index in order to duplicate the corresponding unfactorized row in $\{r\}$. Care is taken during pivoting to avoid the choice, as row l, of a row with a zero or very near to zero corresponding component of $\{x_k\}$. This must be done since, as it is shown in Ref. 14, both the lth row and column must be linearly dependent on the remaining $(n-1)$ ones; thus, we cannot choose a row corresponding to a zero element of the eigenvector.

3) Method III: More straightforwardly, if we think for awhile that $([A]-\lambda_k[I])$ is singular only to the extent of the numerical accuracy with which λ_k is obtained, we can solve Eq. (4) directly along the line adopted for Method II.

Then calling $\bar{\lambda}_k$ the approximate λ_k from Eq. (4), we simply have:

$$\{x_k\}_{,i}=\{v\}\lambda_{k,i}+\{t\} \tag{13a}$$

$$\{x_k\}^T\{x_k\}_{,i}=0 \tag{13b}$$

with

$$\{v\}=([A]-\bar{\lambda}_k[I])^{-l}\{x_k\} \tag{14a}$$

$$\{t\}=-([A]-\bar{\lambda}_k[I])^{-l}[A]_{,i}\{x_k\} \tag{14b}$$

and

$$\lambda_{k,i}=-\frac{\{x_k\}^T\{t\}}{\{x_k\}^T\{v\}} \tag{15}$$

as it is obtained by substituting Eq. (13a) into Eq. (13b).

Recalling Eqs. (6a) and (6b), we can write in an expanded form:

$$([A]-\bar{\lambda}_k[I])^{-l}=\sum_{j\,l}^{n}\frac{1}{\bar{\lambda}_k-\lambda_j}\frac{\{x_j\}\{y_j\}^T}{s_j} \tag{16}$$

assuming now that we are working on a t-decimal-digit computer and that we have computed $\bar{\lambda}_k$ with $r<t$ exact digits and Eq. (6a) is satisfied within the same error, the substitution of Eq. (16) into Eq. (14a), if Eq. (6a) is remembered, gives

$$\{v\}=0(10^r)\{x_k\} \tag{17}$$

where $0(10)^r$ stands for a number of the order of 10^r. In fact, the terms with indexes different from k disappear as they cannot contribute anything within the available precision due to the truncation.

Analogously, for $\{t\}$ we have

$$\{t\}=-0(10^r)\{x_k\}\frac{\{y_k\}^T[A]_{,i}\{x_k\}}{s_k}+\{e\} \tag{18}$$

where

$$\{e\}=\sum_{l\ j\neq k}^{n}\frac{1}{\bar{\lambda}_k-\lambda_j}\{x_j\}\frac{\{y_j\}^T[A]_{,i}\{x_k\}}{s_j} \tag{19}$$

is considered as an error term contributing only to the last $t-r$ digits of $\{t\}$.

The substitution of Eq. (17) and (18) into Eq. (15) gives

$$\lambda_{k,i}\simeq\frac{\{y_k\}^T[A]_{,i}\{x_k\}}{s_k} \tag{20}$$

with a precision of about r digits. Finally, by back-substituting Eqs. (17, 18, and 20) into Eq. (13a), we obtain

$$\{x_k\}_{,i}\simeq\{e\} \tag{21}$$

within approximately $t-r$ digits. Thus, by applying Eqs. (13-15), owing to the quasisingularity of $([A]-\bar{\lambda}_k[I])$, we obtain the approximations of Eqs. (20) and (21).

Now the formulas that give the eigenderivatives exactly when Eq. (2) is assumed, are nothing but Eqs. (20) and (21) themselves with the formal change of $\simeq$ into $=$.[8,9,10,15]

The precision just mentioned is clearly sufficient for $\lambda_{k,i}$ and we can obtain the same for $\{x_k\}_{,i}$ by solving Eqs. (14) with accumulation in double precision. As in this case we add only noise to $\{v\}$ for the digits of order higher than $2r$, which does not change the precision of $\lambda_{k,i}$, but allows the attainment of $\{x_k\}_{,i}$ with r exact digits.

This method requires practically the same amount of calculation as Method II, but is somewhat more compact and can be surely adopted when the eigensolutions are well separated. The application of the method to higher order derivatives is straightforward.

For instance, by differentiating Eq. (4) with respect to the parameter d_l, we can obtain second-order derivatives $\{x_k\}_{,il}$

and $\lambda_{k,il}$ by solving the system:

$$\begin{bmatrix} ([A]-\lambda_k[I]) & \vdots & -\{x_k\} \\ \cdots & \cdots & \cdots \\ -\{x_k\}^T & \vdots & 0 \end{bmatrix} \begin{Bmatrix} \{x_k\}_{,il} \\ \cdots \\ \lambda_{k,il} \end{Bmatrix} = -\begin{Bmatrix} ([A]_{,i}-\lambda_{k,i}[I])\{x_k\}_{,l} \\ +([A]_{,l}-\lambda_{k,l}[I])\{x_k\}_{,i} \\ +[A]_{,il}\{x_k\} \\ \cdots \\ 0 \end{Bmatrix} \quad (22)$$

Thus, in order to obtain the second derivatives, we need the i and l first-order derivatives, but, what is more important, for a given k eigensolution, we always have to solve a system of linear equations, which keeps the same coefficient matrix and differs only in the right-hand side terms. The same occurence can be easily proven for derivatives of any order.

As the coefficient matrix has been factorized once for all, according to one of the three methods illustrated with the algorithm most appropriate to its structure, any derivative of the eigensolution under consideration can be obtained at a minor cost. This is true independently of the particular $[A]$ at hand, as it can be symmetric or unsymmetric, real or complex.

It should be noted that this method remains competitive even in the case in which only the derivative of the eigenvalue is needed, since, except for a symmetric $[A]$, the other methods require the determination of the left eigenvector $\{y_k\}$. Generally the latter is obtained along with $\{x_k\}$ by inverse power iteration,[18] which requires in practice the same work as any of the methods of solution of Eq. (4) presented here.

It is interesting to note that if $\{x_k\}$ is calculated by inverse iteration and if Method III is adoptable because of a good separation, the eigenderivative calculation requires the simple solution of a linear system as the factorization of $([A]-\lambda_k[I])$ is already available.

Application to General Linear and Quadratic Eigenproblems

The most general forms for linear and quadratic eigenproblems are:

$$([A]+\lambda[B])\{x\}=0 \quad (23a)$$

and

$$([A]+\lambda[B]+\lambda^2[C])\{x\}=0 \quad (23b)$$

Often in these cases the normalization is done by the introduction of a weighting matrix $[W]$, which can be one of the matrices appearing in Eqs. (23), or a combination of them. Then Eq. (2) is replaced with:

$$\{x\}^T[W]\{x\}=I \quad (24)$$

$[W]$ can be assumed to be independent from the parameter d_i without loss of generality in any case.

Then, by simply taking the derivatives of Eqs. (23) and (24) with respect to d_i, the formulas equivalent to Eq. (4) for the problems of Eqs. (23) are:

$$\begin{bmatrix} ([A]+\lambda_k[B]) & \vdots & [B]\{x_k\} \\ \cdots & \cdots & \cdots \\ \{x\}^T[W] & \vdots & 0 \end{bmatrix} \begin{Bmatrix} \{x_k\}_{,i} \\ \cdots \\ \lambda_{k,i} \end{Bmatrix} = -\begin{Bmatrix} ([A]_{,i}+\lambda_k[B]_{,i})\{x_k\} \\ \cdots \\ 0 \end{Bmatrix} \quad (25)$$

in the linear problem, and

$$\begin{bmatrix} ([A]+\lambda_k[B]+\lambda_k^2[C]) & \vdots & (2\lambda_k[C]+[B])\{x_k\} \\ \cdots & \cdots & \cdots \\ \{x_k\}^T[W] & \vdots & 0 \end{bmatrix} \times \begin{Bmatrix} \{x_k\}_{,i} \\ \cdots \\ \lambda_{k,i} \end{Bmatrix} = -\begin{Bmatrix} ([A]_{,i}+\lambda_k[B]_{,i}+\lambda_k^2[C]_{,i})\{x_k\} \\ \cdots \\ 0 \end{Bmatrix} \quad (26)$$

in the quadratic problem.

No further comment is needed, since the form of Eqs. (25) and (26) is the same as that of Eq. (4) and all that has been said in the previous paragraph is valid also in these cases; extension to higher-order derivatives is trivial and requires only tedious analytical work to arrive at a formula similar to Eq. (22).

Flutter and Divergence

The most general way of writing the equation expressing the flutter condition is:

$$\left(-\omega^2[M]+j\omega[C]+[K]-\frac{\rho V^2}{2}\left[\mathfrak{A}\left(\frac{\omega b}{V},\mathfrak{M}\right)\right]\right)\{q\}=0 \quad (27)$$

where $j=\sqrt{-1}$, ω is the circular frequency, b is the reference aerodynamic chord, V is the speed of the freestream, $\mathfrak{M}$ is the Mach number $= V/a$, a is the speed of sound corresponding to the altitude density ρ, and $\{q\}$ is the vector of generalized coordinates.

The matrices $[M]$, $[C]$, and $[K]$ are usually called mass, damping, and stiffness matrices; they may be complex as they may include frequency response functions of servos and controls.

Also, the aerodynamic matrix $[\mathfrak{A}]$ is complex and depends on the reduced frequency and Mach number in a transcendental form. As noted in the introduction, Eq. (27) is not properly a usual eigenproblem since, though it is homogeneous in $\{q\}$, the eigenvalue parameters ω and V are not the real and imaginary parts of a complex number, and in any case, do not appear explicitly.

Then, in this case, it is better to forget the eigenproblem and to render unique, regardless of the sign, the determination of $\{q\}$ by adding to Eq. (27) the usual normalizing condition of Eq. (24)

$$\{q\}^T[W]\{q\}=I \quad (28)$$

Equations (27) and (28) are therefore, a truly nonlinear system of $2n+2$ real unknowns, i.e., ω, V, $\mathrm{Re}\{q\}$, and $\mathrm{Im}\{q\}$, where Re is the real part and Im the imaginary one of a complex number.

In fact, this affords an extremely efficient method, and one of the very few[17-19] to solve the flutter equation directly without tracking damped solutions at various speeds.

If we call $[F]$ the complex matrix of the coefficients of Eq. (27) so that $[F]\{q\}=0$, in order to calculate the first derivatives after differentiating Eqs. (27) and (28) term by term, we solve the following real system of $2n+2$ real equations

$$\begin{bmatrix} \mathrm{Re}[F] & -\mathrm{Im}[F] & \mathrm{Re}\{w\} & \mathrm{Re}\{z\} \\ \mathrm{Im}[F] & \mathrm{Re}[F] & \mathrm{Im}\{w\} & \mathrm{Im}\{z\} \\ \mathrm{Re}\{q\}^T[W] & -\mathrm{Im}\{q\}^T[W] & 0 & 0 \\ \mathrm{Im}\{q\}^T[W] & \mathrm{Re}\{q\}^T[W] & 0 & 0 \end{bmatrix} \times \begin{Bmatrix} \mathrm{Re}\{q\}_{,i} \\ \mathrm{Im}\{q\}_{,i} \\ \omega_{,i} \\ V_{,i} \end{Bmatrix} = \begin{Bmatrix} \mathrm{Re}\{r\} \\ \mathrm{Im}\{r\} \\ 0 \\ 0 \end{Bmatrix} \quad (29)$$

with

$$\{w\} = \left(-2\omega[M] + j[C] - \frac{\rho V b}{2}\frac{\partial[\mathfrak{A}]}{\partial(\omega b/V)}\right)\{q\} \tag{30a}$$

$$\{z\} = \left(-\frac{\rho V^2}{2a}\frac{\partial[\mathfrak{A}]}{\partial\mathfrak{M}} - \rho V[\mathfrak{A}] + \frac{\rho\omega b}{2}\frac{\partial[\mathfrak{A}]}{\partial(\omega b/V)}\right)\{q\} \tag{30b}$$

$$\{r\} = \left(\omega^2[M]_{,i} - j\omega[C]_{,i} - [K]_{,i} + \frac{\rho V^2}{2}[\mathfrak{A}]_{,i}\right)\{q\} \tag{30c}$$

Equation (29) can be solved directly with real arithmetic, since it is, in general, not a large system of equations.

In any case, by using Methods II and III, we can render the application even less expensive, as explained in Refs. 16 and 17.

Generally, only $\omega_{,i}$ and $V_{,i}$ are needed and the methods available for their calculation always need[20-22] the right and left eigenvectors of Eq. (27). The latter is normally calculated by inverse iteration. Thus, the work equivalent to the solution of Eq. (29) always has to be done.

If we use Eqs. (27) and (28) as a nonlinear system for evaluating the flutter solution and we do this by the Newton-Raphson method, in each iteration we solve linear systems with the same coefficient matrix but with different right-hand sides. Therefore, when we come to convergence, the evaluation of the flutter derivatives simply requires another solution with $\{r\}$ as the known right-hand side. The aeroelastic divergence equation comes naturally from Eq. (27) when $\omega = 0$, so we can write

$$\left(-\frac{\rho V^2}{2}[\mathfrak{A}(\mathfrak{M})] + [K]\right)\{q\} = 0 \tag{31}$$

where $[\mathfrak{A}]$ and $[K]$ are now real quantities.

This can be assumed as a linear eigenproblem with $\rho V^2/2$ as an eigenvalue only by holding $\mathfrak{M}$ constant; but, in this case, we have to also solve Eq. (31) for different ρ in order to cover all the altitudes of the flight envelope.

On the contrary, if we want to match a critical value of V for a given altitude, the unknown appears in $[\mathfrak{A}]$ through $\mathfrak{M}$ and we have a nonlinear eigenproblem which is analogous to Eq. (27).

First derivatives of $\{q\}$ and V can be obtained from:

$$\left[\begin{array}{c|c} \left([K] - \frac{\rho V^2}{2}[\mathfrak{A}]\right) & -\left(\rho V[\mathfrak{A}] + \frac{\rho V^2}{2a}\frac{\partial[\mathfrak{A}]}{\partial\mathfrak{M}}\right) \\ \hline \{q\}^T[W] & 0 \end{array}\right] \left\{\begin{array}{c} \{q\}_{,i} \\ \hline V_{,i} \end{array}\right\} = \left\{\begin{array}{c} \left(\frac{\rho V^2}{2}[\mathfrak{A}]_{,i} - [K]_{,i}\right)\{q\} \\ \hline 0 \end{array}\right\} \tag{32}$$

The formulas of higher-order derivatives are not presented here, since they will add nothing to the content of the method and can be easily developed by further differentiating Eqs. (29) and (31).

Conclusions

The procedure presented here allows efficient calculation of eigenderivatives for a large class of eigenproblems that may be found in practical response and stability analyses, provided they possess a full set of distinct eigenvalues.

It requires no further data but the eigensolution at hand and allows the determination of eigenderivatives of an unsymmetric system without the solution of the transposed eigenproblem.

Full exploitation of the possible sparseness and symmetry of matrices is allowed so that derivatives of large problems can be calculated efficiently and with acceptable precision.

The method always allows a unified treatment in the calculation of derivatives in aeroelastic stability problems for the solution of which it can be considered a natural appendix of the techniques that determine critical solutions directly.

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